On Ducks and Bathtubs

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Following a recent exchange of messages in the Usenet newsgroup sci.math, Prof. Mückenheim kindly agreed to show you this rebuttal of his opinion of what set theory says about following scenario:

> Scrooge McDuck every day gets \$1000 and returns \$1. If he happens to return the right dollars, he will get unmeasurably rich. If he happens to return the wrong dollars, he will go bancrupt. This is an unscientific result. It is the result which set theory is based upon. Do you think that anybody will accept this latter interpretation?
>
> Note that Duck's wealth is not determined by $\lim card = \aleph_0$ but by $card \lim = 0$. His wealth is the cardinality of his "final" set $\{\}$ which is 0.[1]

Every day, McDuck is some \$1 bills from a bank with limitless resources. Every day, he gives some notes back which the back destroys, so that he is never given the same bill twice. This is the first hint that something is wrong. Wealth does not depend on the particular notes you hold, but it's central to that accusation that Prof. Mückenheim makes of set theory.

To keep things simple, let's number the notes $1, 2, 3, \ldots$ and we'll give McDuck only \$2 a day. (Any number of notes > 1 will do). McDuck's wealth, on any particular day, is determined by the *size* of the set of notes that he holds so which *specific* notes they are is not important. If he always returns lowest numbered note, the sequence

$$L_n = \{2\}, \{3, 4\}, \{4, 5, 6\}, \{5, 6, 7, 8\}, \{6, 7, 8, 9, 10\}, \ldots$$

represents the notes he holds on consecutive days. If, instead, he chooses to return the *highest* numbered note instead, he will end up with all the odd numbered notes:

$$H_n = \{1\}, \{1, 3\}, \{1, 3, 5\}, \{1, 3, 5, 7\}, \{1, 3, 5, 7, 9\}, \ldots$$

There are, of course, lots of other possibilities, but they all show McDuck with the same *number* of notes on each day. It's sensible, then, to use this number—the cardinality of the set—to show McDucks's wealth, w_n on day n. Since $w_n = |L_n| = |H_n| = n$. the limits of these numerical sequences

$$\lim w_n = \lim |L_n| = \lim |H_n| = \lim n = \aleph_0 \quad \text{(or } \infty \text{ if you prefer).}$$

all agree on McDuck's happy future.

But Prof. Mückenheim explicitly tell us that we should *not* use the limit of the cardinalities as the measure of long-term wealth. Instead he says we should use the cardinalities of the limits

$$|\lim L_n| = 0 \quad \text{and} \quad |\lim H_n| = \aleph_0.$$

[1] This quote is from a later, summary, post:
https://groups.google.com/d/msg/sci.math/ootYk7_UKlU/xLv5dft5nrAJ

which are *not* the same for the two sequences. Let's take a moment to see where these results come from.

Some sequences of sets do, in fact, have a limit set. The basic idea comes from the phrase I used earlier: the limit is the set of notes McDuck "ends up with". A more rigorous definition (which I've included at the end so you can see the details) corresponds to those notes that McDuck gets given and *then keeps indefinitely.* In the case of the sequence H_n it is reasonable to say that McDuck "ends up with" note number 5 on day 3, because we know he will never give it back. But in the case of L_n we would *not* say that he "ends up with" note 5 (despite getting it on day 3) because we know he will hand it back two days later. Keep in mind, though, that this phrase is only a hint. It's just a shorthand for the notes that are kept indefinitely.

For the sequence H_n, the limit is the set of odd numbers; McDuck never gives back an odd numbered note so he holds all of then indefinitely. But in the case of L_n, no note is held indefinitely, so the limit set is empty. In fact, McDuck can make the limit any size he likes, and there are (literally) countless limit sets that are possible depending on what McDuck does each day.[2]

At first sight it seems paradoxical that McDuck gets richer and richer, despite the fact that he will will, eventually, hand back every note. However there are plenty of familiar situations where something similar happens. Think, for example of water flowing into a bathtub. The tub can fill up even if the plug is out, provided the water flows in faster than it drains way. But what is the fate of the individual water molecules? Every molecule that flows in to the tub is destined to leave it eventually, but the later they arrive, the longer it takes for them to reach the drain. This ever-increasing transit time, together with the endless flow of water, is what permits the bath to keep on filling indefinitely.

But not all molecules need to leave. If the incoming water hits a wall that dives that bath in two, only those molecules that happen to fall on the side with the drain will be lost forever. In both cases (given equal flows) the amount of water in the tub continues to increase at the same rate. Yet in one case the number of molecules that will be kept indefinitely is zero, and in the other it increases without bound. Prof. Mückenheim is telling you to consider the eventual fate of the individual molecules and to ignore the ever-increasing volume of water in the tub.

I hope it is clear by now what's wrong with the opening quote. Prof. Mückenheim is simply telling you to use the wrong limit for the job. McDuck's wealth is not determined by which notes he hold indefinitely, but by how many he holds. Set theory's definition of the limit does say something about the notes held, but not what Prof. Mückenheim would like you to think it says. The only unscientific thing here is using the wrong mathematical tool.

But set theory is not off the hook yet. Prof. Mückenheim uses the phrase "final set" to prompt you to think, informally, of a limit as the "result" of some sequence. You are not meant to take it literally (that's why the word "final" is in so-called "scare quotes") but rather you are supposed to see the sets in L_n getting bigger and bigger, and be shocked that set theory says that the limit (the supposed "result" or "final" set) is empty. But this surprise just replies on things you don't yet know about limits.

It turns out that the seemingly odd limits of set theory are directly analogous to the limits used in analysis. They simply reply on necessarily very different notions of order and distance. I would like to be able show you the details, because that would take the mystery away, but that would need many pages. Though the limit of L_n is, at first sight, surprising, that is not a good enough reason to rejects it, and it's certainly not a reason to suggest that it represents anything "final" about McDuck's wealth.

[2] It's helpful to take a moment and work out what limit sets are possible, and which ones are not.

Extra: A more formal definition of the set limit

Whilst I can't describe how set limits relate to the more usual analytic limits, I must give you a more formal definition of the limit set so that you can see how they are calculated. Perhaps the simplest definition is

$$\lim_{n\to\infty} S_n = \{x \mid \lim_{i\to\infty} I_x(i) = 1\}$$

which says that that the limit set consists of those elements whose *indicator function* tends to the limit 1. The limit set exists provided all of the numerical limits exist. The indicator function for an element x, is just

$$I_x(i) = \begin{cases} 1 & \text{if } x \in S_i \\ 0 & \text{if } x \notin S_i \end{cases}$$

This is all much simpler than it looks, as you can see from this table showing the first few values of some of the indicator functions for the set sequence L defined a couple of pages ago.

$i =$	1	2	3	4	5	6
$L_i =$	$\{2\}$	$\{3,4\}$	$\{4,5,6\}$	$\{5,6,7,8\}$	$\{6,7,8,9,10\}$	$\{7,8,9,10,11,12\}$
$I_1(i) =$	0	0	0	0	0	0
$I_2(i) =$	1	0	0	0	0	0
$I_3(i) =$	0	1	0	0	0	0
$I_4(i) =$	0	1	1	0	0	0
$I_5(i) =$	0	0	1	1	0	0
$I_6(i) =$	0	0	1	1	1	0

See, for example, that the function $I_4(i)$ *indiactes* which sets contain 4 by having the value 1 whenever 4 is in S_i.

Because we know the rule that generates this set sequence, we can show that $I_x(i)$ eventually becomes 0, and remains 0, for every x. Since therefore $\lim_{i\to\infty} I_x(i) = 0$ for every x the limit set is the empty set.

And here is a similar table for the set sequence H.

$i =$	1	2	3	4	5	6
$T_i =$	$\{1\}$	$\{1,3\}$	$\{1,3,5\}$	$\{1,3,5,7\}$	$\{1,3,5,7,9\}$	$\{1,3,5,7,9,11\}$
$I_1(i) =$	1	1	1	1	1	1
$I_2(i) =$	0	0	0	0	0	0
$I_3(i) =$	0	1	1	1	1	1
$I_4(i) =$	0	0	0	0	0	0
$I_5(i) =$	0	0	1	1	1	1
$I_6(i) =$	0	0	0	0	0	0

This time, the sets contain only odd numbers, but every odd number is present indefinitely. Thus $I_x(i)$, for odd x, eventually becomes 1 and stays 1, and we have

$$\lim_{i\to\infty} I_x(i) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

The set sequence limit is therefore $\lim_{n\to\infty} T_n = \{1, 3, 5, \ldots\}$.